

Loss Rate Inference in Multi-Sources and Multicast-Based General Topologies

Weiping Zhu, *member, IEEE*

Abstract—Loss tomography has received considerable attention in recent years and a number of estimators have been proposed. Unfortunately, almost all of them are devoted to the tree topology despite the general topology is more common in practice. In addition, most of the works presented in the literature rely on iterative approximation to search for the maximum of a likelihood function formed from observations, which have been known neither scalable nor efficient. In contrast to the tree topology, there is few paper dedicated to the general topology because of the lack of understanding the impacts created by the probes sent by different sources. We in this paper present the analytical results obtained recently for the general topology that show the correlation created by the probes sent by multiple sources to a node located in an intersection of multiple trees. The correlation is expressed by a set of polynomials of the pass rates of the paths connecting the sources to the node. In addition to the expression, a closed form solution is proposed to obtain the MLE of the pass rates of the paths connecting the sources to the node. Then, two strategies are proposed to estimate the loss rate of a link for the general topology: one is path-based and the other is link-based, depending on whether we need to obtain the pass rate of a path first. The two strategies are compared in the context of the general topology that shows each has its advantages and the link-based one is more general. Apart from proving the estimates obtained are the MLEs, we prove the estimator presented here has the optimal asymptotic property.

Index Terms—Decomposition, General topology, Link-based Estimator, Path-based Estimator, Loss tomography, Tree topology.

I. INTRODUCTION

Network tomography was proposed in [1] that suggests the use of end-to-end measurement and statistical inference to estimate network characteristics of a large network. Since then, a considerable amount of works has been done in this area and the works reported so far have covered almost all of the characteristics that were previously obtained by direct measurement for a small network. The works can be classified into loss tomography [2], delay tomography [3], [4], [5], [6], [7], loss pattern tomography [8], source-destination traffic matrix [9] and shared congestion flows [10]. Despite such a overwhelming enthusiasm and a wealth of publications, most of the works in this area are at the preliminary stage, and most of the estimators or algorithms proposed for estimations aim at proof of concept while the efficiency and scalability of them are very much overlooked. For instance, loss tomography has been studied for a decade and a number of algorithms have been published to estimate link-level loss rates from end-to-end measurement. Unfortunately, almost all of the algorithms are limited into the tree topology and most of them resort on

iterative approximation. To overcome this problem, we in this paper propose a number of maximum likelihood estimators to estimate link-level loss rate of a general network, all of them are in closed form.

The topologies used to connect sources to receivers can be divided into two classes: tree and general, which leads to two types of estimators, one for a topology. As stated, the tree one has received much more attention than its counterpart. Almost all of the estimators proposed previously target the tree topology because of the simplicity of the correlation embedded into the observations of the receivers attached to a tree topology. Despite the large amount of works, there are only a few of them providing analytical solutions as [11] [12] [13]. That leads to the use of iterative procedures to approximate the root of a likelihood function although knowing such a procedure is neither efficient nor scalable [14]. In contrast to the tree topology, there has been little systematically study in both theory and experiments for the general topology despite the general topology is more common than the tree one in practice. As a result, there are only a handful of estimators that have been proposed for the general topology. To improve this, we in this paper provide our findings about the impact of multiple sources on loss estimation that lead to a number of maximum likelihood estimators for the general topology, including an analytical solution, and two divide-and-conquer ones. All of them completely eliminate iterative procedure involved in the estimation. Compared with the estimators proposed previously for the general topology [11], [15], the estimators proposed here not only perform better, but also extend our understanding of loss tomography from the tree topology to the general one. In fact, the results presented in this paper generalize those obtained from the tree topology since the tree is a special case of the general topology. Further, the results may be extended into other tomographies, such as delay, in the general topology.

A. Related Works

Multicast Inference of Network Characters (MINC) is the pioneer of using multicast probes to create correlated observations at the receivers of the tree topology, where a Bernoulli model is used to model the losses occurred on a link. Using this model, the authors of [11] derive a direct expression of the pass rate of a path connecting the source to an internal node. The expression is in the form of a polynomial [11], [16], [17]. To ease the concern of unsalability of using numeric method to solve higher degree polynomials (> 5), Duffield *et. al.* propose an explicit estimator in [12] for the tree topology. Although the estimator has the same asymptotic variance as

the MLE to first order, it is not an MLE. When $n < \infty$, the estimate obtained by the estimator can be very different from an MLE. In contrast, [11] proposed an iterative approach to search for the maximum of the general topology. Later, Bu *et al.* attempted to use the same model as the one used in [11] for the general topology, but could not derive a direct expression as its predecessor [15]. Without analytic results, the authors of [15] then resort on iterative procedures, e.g. the EM, to approximate the maximum of the likelihood function. In addition, a minimum variance weighted average (MVWA) estimator is proposed as an alternative in comparison with the EM one. The MVWA in fact is a variant of the minimum chi-square estimator [18]. The experimental results confirm this problem [15], where the EM outperforms the MVWA when the sample size is small. Nevertheless, there is no proof that the solution obtained by the EM algorithm is the MLE since the iterative procedures, e.g. EM algorithm, can converge to a local maximum unless the solution space is proved to be strictly concave.

Considering the unavailability of multicast in some networks, Harfoush *et al.* and Coates *et al.* independently proposed the use of the unicast-based multicast to discover link-level characteristics [19], [20], where Coates *et al.* also suggested the use of the EM algorithm to estimate link-level loss rates that may not be scalable as previously stated.

To improve the scalability of estimation, Zhu and Geng propose a bottom up algorithm in [21]. The algorithm is independent of topology, i.e. it can be applied to the general topology as well [22]. The estimator, rather than estimating the loss rates of all links together as an iterative procedure, adopts a step by step approach to estimate the loss rate of a link one at a time and from bottom up. At each step the algorithm uses a closed form formula to estimate the loss rate of a link. Despite the effectiveness, scalability and extensibility to the general topology, the estimator is not the MLE as the one presented in [12] because the statistics used in estimation are not sufficient. Recently, the authors of [23] use a divide-and-conquer approach to tackle the general topology, which divides a general network into a number of independent trees at the roots of intersections. It uses a fixed point algorithm to estimate the number of probes reaching the roots of intersections. The fixed point algorithm is based on an iterative procedure to achieve its goal, although experimental studies show it has a fast convergence rate since the solution space is strictly concave.

B. Contribution and Paper Organization

As stated, there has been a lack of understanding the impact of probes sent from multiple sources to a link or a node in the general topology. In addition, iterative procedures are still used in all estimators proposed for the general topology [23]. We are aiming to improve the unsatisfied situation and present the latest findings in this paper that contribute to loss tomography in the following respects:

- 1) we derive a set of polynomials to express the pass rate of the paths connecting a source to a node in the general topology. The derivation shows the pass rate of a path

in the general topology is no longer independent from each other in contrast to that in a tree topology. This finding generalizes the results of [11] and is applicable to various topologies.

- 2) we derive a closed form MLE solution to the set of the polynomials of related paths. The solution shows that the MLE considers the probes sent from all related sources that generalizes the result obtained from the tree topology.
- 3) we present a divide-and-conquer strategy to break up a general network into a number of independent trees to avoid the necessity of solving a large number of polynomials. Two algorithms are proposed subsequently to the independent trees, one for the descendants of a decomposing point, the other for the ancestors of the decomposing point. The algorithms previously developed for the tree topology can be applied to the former with little help while the latter requires its own algorithm.
- 4) we propose two estimators: link-based and path-based, for the ancestors of a decomposing point. The result shows each of the estimators has its own advantages and disadvantages. The link-based one is more general than the path-based one.
- 5) we clarify some misconceptions about using multiple sources, in particular to those focusing on the complexity created by multiple sources in estimation but ignoring the benefits brought by the multiple sources on data consistency, variance reduction, and convergence rate.

The rest of the paper is organized as follows. In Section 2 we present the essential background, including the notations and statistics used in this paper. In Section 3, we present our latest findings and contributions to the loss inference of the general topology. Section 4 provides the detail description of the three estimators, plus some asymptotical properties of the estimator proposed in this paper. Section 5 is devoted to the statistical properties of the estimators presented in this paper. Simulations are presented in Section 6 that shows the estimates of overlapped links are converged quicker than other links. The last section is devoted to concluding remark.

II. NOTATIONS AND STATISTICS

A general topology differs from a tree one in a number of aspects. The most important one is the use of multiple trees to cover a general network, which leads some parts of the network covered more than once by the trees. Those parts are called intersection areas, or simply intersections. If a source is assigned to the root of a tree to send probes to the receivers attached to the tree, the receivers attached to an intersection can observe the probes sent from multiple sources. Then, the question is whether we need to consider the probes sent by different sources in estimation. If not, the estimate obtained from a source can be different from that obtained from another source and the difference can be noticeable unless the number of probes sent by each source goes to infinite that is not appropriate in practice. If yes, we must determine the contribution of each source on the estimate. In addition, the 1-to-1 mapping between nodes and links held

in the tree topology no longer exists in the general topology since some nodes can have more than one parents.

A. Notation

A multicast tree or subtree used to connect the source to receivers is slightly different from an ordinary tree at the root that has only a single child. The link connecting the root node to the child is called the root link of the tree. Let \mathcal{N} denote a general network consisting of k trees, where $S = \{s^1, s^2, \dots, s^k\}$ denote the sources attached to the trees. In addition, V and E denote the nodes and links of \mathcal{N} . Each node is assigned a unique number, so is a link. $|V|$ and $|E|$ denote the number of nodes and the number of link of the network, respectively. The two nodes connected by a link are called the parent node and the child node of the link, where the parent receives probes from its parent and forward them to its children. In addition, $p(s, i)$ denotes the parent link of link i toward source s . The nodes that have no parent(s) are the roots of the multicast trees. Each of the multicast trees and the subtrees is named after the number assigned to its root link, where $T(i), i \in E$ denotes the subtree with link i as its root link. Further, let $R(s), s \in S$ denote the receivers attached to the multicast tree rooted at s and let $Rs(i), i \in E$ denotes the receivers attached to the multicast subtree rooted at link i . The largest intersection between two trees is called the *intersection* of the trees and the root of an intersection is called the *joint node* of the intersection. Note that an intersection is an ordinary tree but a multicast one. We use J to denote all joint nodes and use $S(j), j \in V \setminus S$ to refer the sources that send probes to node j . If $f_1^j(i)$, simply $f^j(i)$ later, is used to denote the parent of node i on the way to source s^j and $f_l^j(i)$ to denote the ancestor that is l hops away from node i on the path to source s^j , we have $f_k^j(i) = f^j(f_{k-1}^j(i))$. Further, let $a(j, i) = \{f^j(i), f_2^j(i), \dots, f_k^j(i)\}$, where $f_{k+1}^j(i) = s^j$, denote the ancestors of node i to s^j . Let $a(i) = \{a(j, i), j \in S(i)\}$ denote all the ancestors of node i . In addition, except for leaf nodes each node has a number of children, where d_i denotes the children of node i and $|d_i|$ denotes the number of children of node i .

If $n^s, s \in S$ denotes the number of probes sent by source s , each probe $o = 1, \dots, n^s$ gives rise of an independent realization $X^s(o)$ of the probe process X , $X_k^s(o) = 1, k \in E$ if probe o passing link k ; otherwise $X_k^s(o) = 0$. The observation of $\Omega = \{\Omega_s, s \in S\}, \Omega_s = (X^s(o)), o = 1, 2, \dots, n^s$ comprise the data set for inference. In addition, let $Y_k^s(i), i = 1, 2, \dots, n^s, k \in E, s \in S$ denote the state of link k obtained by examining Ω_s for probe i to see whether it reaches at least one of $Rs(k)$. $Y_k^s(i) = 1$ if the probe reaches, otherwise $Y_k^s(i) = 0$.

B. Sufficient Statistic

We assume that each source sends probes independently, and the observations of the arrivals of probes at nodes and receivers are also assumed independent. Then, the loss process of a link is considered an independent identical distributed (*i.i.d.*) process. This also makes the collective impacts of probes sent by the sources to a link *i.i.d.*. Further, the likelihood function

of an experiment takes either a product form of the individual likelihood function or a summation form of the individual log-likelihood function. Therefore, we will consider a single source first in the following discussion, and then add a number of the single ones together to form a multiple-sources general log-likelihood function.

If a probe sent by source s reaches $Rs(i)$, the probe must pass link i . Then, by examining the observations of $Rs(i)$, we are able to confirm some of the probes sent by the source passing link i . Based on the confirmed passes of each link, we have a set of sufficient statistics to write the likelihood function of Ω . To obtain the confirmed passes of link i , we must examine each observation received by $Rs(i), i \in E$. For each $i \in E$, let

$$Y_i^s(j) = \max_k X_k^s(j), k \in Rs(i).$$

If $Y_{c(i)}^s(j) = 1$ probe j reaches at least one of $Rs(i)$, that also implies it reaching node i . Further, considering $Y_i^s(j)$ and $Y_{p(s,i)}^s(j)$, we can confirm some of the probes sent by s passing link i . If

- 1) $Y_i^s(j) = Y_{p(s,i)}^s(j) = 1$, it is confirmed that probe j passes link i ; or
- 2) $Y_i^s(j) = 0$ and $Y_{p(s,i)}^s(j) = 1$, it is confirmed that probe j passes link $p(s, i)$ but uncertain whether the probe passes link i ; or
- 3) $Y_i^s(j) = Y_{p(s,i)}^s(j) = 0$, it is obvious since uncertainty is transferable from a node to its descendants.

If the first scenario occurs, we need to have $(1 - \theta_i)$ in the likelihood function. If the second one occurs, we need to have $(\theta_i + (1 - \theta_i)(1 - \beta_i))$ in the likelihood function, where $(1 - \beta_i) = P(\bigwedge_{r \in R(i)} X_r = 0 | X_i = 1; \theta)$ is the loss rate of subtree i that considers all possible combinations that lead to $\bigwedge_{r \in R(i)} X_r = 0$ since uncertainty is transferable. If the last one occurs, we do not need adding any thing into the likelihood function because its contribution to the likelihood function is considered by one of its ancestors that has the second scenarios for this probe.

Considering all probes sent by source s , and let

$$n_i(s, 1) = \sum_{j=1}^{n^s} Y_i^s(j),$$

denote the number of the confirmed passes of the probes sent by source s on link i . In addition, let

$$n_i(s, 0) = n_{p(s,i)}(s, 1) - n_i(s, 1)$$

be the number of probes sent by source s that turn to uncertain in $T(i)$. Further, considering all sources, we have

$$n_i(1) = \sum_{s \in S(i)} n_i(s, 1) \quad n_i(0) = \sum_{s \in S(i)} n_i(s, 0)$$

be the total number of probes confirmed from observations that pass link i and the total number of probes that turn to uncertain in $T(i)$. $n_i(1), i \in E$ consist of a set of sufficient statistics [23].

III. MAXIMUM LIKELIHOOD ESTIMATOR

The statistics presented in the previous section are extended from those presented in [23]. In [23], we also attempted to tackle the general topology, but did not derive a closed form solution. Instead, we came up with a strategy to decompose a general topology into a number of independent subtrees. Nevertheless, that experience inspires us to further investigate the correlation in Ω , where an insight is discovered from this investigation. For completeness, the link-based estimator is presented first. Then, we present the insight which finally leads to a path-based estimator.

A. Link-based Estimator

Given the set of sufficient statistics, we can write the log-likelihood function of Ω . Using the same strategy as [24], the log-likelihood function of a general network is presented as follows:

$$L(\theta) = \sum_{i \in E} [n_i(1) \cdot \log(1 - \theta_i) + n_i(0) \cdot \log \xi_i] \quad (1)$$

where $\xi_i = \theta_i + (1 - \theta_i)(1 - \beta_i)$. Differentiating $L(\theta)$ with respect to (wrt.) θ_i and setting the derivatives to 0, we have a set of equations. Solving them, we have the followings:

$$\theta_i = \begin{cases} 1 - \frac{n_i(S(i), 1)}{\beta_i}, & i \in RL, \\ 1 - \frac{n_i(S(i), 1)}{n_{f(i)}(S(i), 1) + \text{imp}(S(i), f(i))}, & i \in SBRL, \\ 1 - \frac{\sum_{j \in S(i)} n_i(j, 1)}{\sum_{j \in S(i)} [n_{f(i)}(j, 1) + \text{imp}(j, f(i))]}, & i \in SSNL, \\ \frac{\sum_{j \in S(i)} [n_i(j, 0) + \text{imp}(j, f(i))]}{\sum_{j \in S(i)} [n_{f(i)}(j, 1) + \text{imp}(j, f(i))]}, & i \in AOL, \end{cases}$$

where RL denotes root links, SBRL denotes the links that are not root link but only receive probes from a single source, SSNL denotes the links in intersections but not leave, and AOL denotes all others, i.e. leaf links. Differentiating $L(\theta)$ wrt. θ_i and setting the derivatives to 0. Then, we have a set of equations as follows, one for a group of links. Let $\beta_i = 1, i \in AOL$ and $\text{imp}(j, i)$ denotes the impact of $n_k(j, 0), k \in a(j, i)$, on the loss rate of link i .

$$\text{imp}(j, i) = \frac{n_k(j, 0) \cdot p_{a_i}(k) \cdot \xi_i \cdot \prod_{\substack{l \in a(i) \\ l \geq k}} \prod_{q \in C_l \setminus l} \xi_q}{\theta_k + (1 - \theta_k) \prod_{q \in C_k} \xi_q}$$

where

$$p_{a_i}(k) = \prod_{\substack{l \in a(j, i) \\ l \geq k}} (1 - \theta_l).$$

Despite the estimate obtained by (2) has been proved to be the MLE in [23], we have not found an analytical solution as

the one presented in [23] for the general topology since (2) is a transcendental equation in the form of $\theta_i = f(\Theta)$, where Θ is the set of the parameters to be estimated, one for a link in E . To overcome this, we proposed a fixed-point algorithm to approximate the number of probes reaching a joint node in [23], and then a general network can be decomposed into a number of independent trees in order to use the estimator developed for the tree topology.

B. Insight and Remark

Despite there is no a closed form solution to (2), the simple and uniform appearance of the four equations provides such an insight of loss tomography that can be expressed in the following remark:

Remark: regardless of the topology and the number of sources, the MLE of the loss rate of a link can be obtained if we know:

- 1) the total number of probes reaching the parent node of the link, e.g. n_i , $n_{f(i)}(S(i), 1) + \text{imp}(S(i), f(i))$, or $\sum_{j \in S(i)} [n_{f(i)}(j, 1) + \text{imp}(j, f(i))]$ of (2);
- 2) the total number of probes reaching the receivers via the link, e.g. $n_i(S(i), 1)$, or $\sum_{j \in S(i)} n_i(j, 1)$ of (2); and,
- 3) the pass rate of the subtree rooted at the child node of the link, e.g. β_i of (2).

Note that the link stated in the remark can be a path consisting of a set of links serially connected. For the tree topology, there is only one source, 1) and 2) can be suppressed to the pass rate of the path connecting the source node to an internal node. Then, a polynomial formula as the one presented in [11], i.e.

$$(2) \quad H_i(A_i, \gamma) = 1 - \frac{\gamma_i}{A_i} - \prod_{j \in d_i} (1 - \frac{\gamma_j}{A_i}) = 0, \quad (3)$$

is obtained that expresses MLE of the path from the source to node i . In fact, (3) expresses $1 - \beta_i$, the loss rate of subtree i , in two different forms of function of A_i : the product of the loss rates of the multicast subtrees rooted at node i and $1 - \frac{\gamma_i}{A_i}$. Then, how to solve a high degree polynomial becomes the key obstacle blocking a closed form solution to the tree topology. The problem has been solved by [23] where an equivalent transformation is proposed that merges multiple descendants into two virtual ones.

Using the same strategy as [23] and the set of sufficient statistics, (2) shows the impact of multiple sources on a link, regardless the link to be estimated is in an intersection or not. The result, on one hand, shows that as the tree topology, the cumulative impact is built from $n_j(s, 0), j \in a(s, i)$ to link i ; on the other hand, it shows that the impact comes from multiple sources, i.e. $\text{imp}(j, i), j \in S(i)$, in a general topology that makes a closed form estimator almost impossible. This is because (2) is a multivariate polynomial relating to all links in a general topology. To find a solution, we need to solve a set of multivariate polynomials, one for a link. That is intractable at least at this moment except for using approximation.

C. Path-based Estimator

There has been no path-based likelihood function for the general topology as far as we know. This is partially due

to the lack of sufficient statistics of a link or a path from observations. With the help of the minimal sufficient statistics proposed in Section 2, we are able to write a path-based likelihood function here, and then derive a set of likelihood equations, and finally have a path-based estimator that provides MLE for the general topology.

Because of the intractability of the link-based estimator, our attention is switched to a path-based estimator although the number of paths is the same as the number of links. The path-based estimator differs from the link-based one, based on the remark, in the following aspects:

- we know the total number of probes sent by a source; and
- we know the total number of probes reaching $Rs(i)$ and know the sources of the arrived probes.

As the tree topology, β_i is unknown and can be explicitly expressed by the available information. The difference between the estimator of the tree topology and that of the general one is at the intersections of trees in the general topology, which makes the trees intersected dependent. Then, we need to determine the correlation between the trees and the number of paths involved in an intersection.

Let

$$A(s, i) = \prod_{j \in a(s, i)} (1 - \theta_j), \quad \forall i \in V, \text{ and } \forall s \in S(i) \quad (4)$$

be the pass rate of the path from source s to node i . It is easy to prove that (4) is the bijection, Γ , from Θ to A , where Θ is the support space of $\{\theta_i, i \in E\}$ and A is the support space of $\{A(s, i), i \in V \setminus S, s \in S(i)\}$. The statistics, $n_i(s, 1), i \in V, s \in S(i)$, are still the sufficient statistics for the path-based likelihood function since the number of probes sent by s and observed by $Rs(i)$ is the confirmed number of probes passing the path connecting s to i . Then, we have the following theorem for the MLE of a path in the general topology.

Theorem 1. *The likelihood equation describing the pass rate of a path connecting source s to node i can be expressed as*

$$A(s, i) = \frac{\gamma_i(s)}{\beta_i}, \quad s \in S(i). \quad (5)$$

where the empirical $\hat{\gamma}_i(s) = \frac{n_i(s, 1)}{n^s}$

Proof: Using the sufficient statistics, we can write the path-based likelihood function as

$$\begin{aligned} P(A(s, i)) &= \prod_{i \in V \setminus S} \prod_{s \in S(i)} \left[A(s, i)^{n_i(s, 1)} (1 - A(s, i) + A(s, i)(1 - \beta_i))^{(n_s(1) - n_i(s, 1))} \right] \\ &= \prod_{i \in V \setminus S} \prod_{s \in S(i)} \left[A(s, i)^{n_i(s, 1)} \times (1 - A(s, i)\beta_i)^{(n_s(1) - n_i(s, 1))} \right] \end{aligned}$$

Changing it to the log-likelihood function, we have

$$\begin{aligned} L(P(A(s, i))) &= \sum_{i \in V \setminus S} \sum_{s \in S(i)} \left[n_i(s, 1) \log A(s, i) \right. \\ &\quad \left. + (n_s(1) - n_i(s, 1)) \log(1 - A(s, i)\beta_i) \right]. \end{aligned}$$

Differentiating it wrt $A(s, i)$ and let the derivative be 0, we have

$$A(s, i) = \frac{\gamma_i(s)}{\beta_i}, \quad s \in S(i).$$

(5) is identical to that obtained for the tree topology except for having a condition for $s \in S(i)$. We call it the consistent condition because it states the necessity of a consistent β_i for those trees intersected at node i . This condition states that an MLE estimator must consider all probes sent from $S(i)$ to i . Compared with the link-based estimator, the path-based estimator only considers the intersected paths during estimation that substantially reduces the correlation involved in estimation. To obtain the MLE $\hat{A}(s, i)$ from the set of equations defined by (5), we need:

- to obtain a consistent β_i for all $\gamma_i(s), s \in S(i)$ and to express β_i in two different ways as the functions of $A(s, i)$,
- to link the two expressions by an equation, and to prove there is a unique solution to the equation.

The following two theorems address them, respectively.

Theorem 2. *Let $A(k, i)$ be the pass rate of the path connecting $k, k \in S(i)$ to $i, i \in V \setminus S$ in a network of the general topology. There is a polynomial, $H(A(k, i), S(i))$, as follows to express the estimate of $A(k, i)$.*

$$\begin{aligned} H(A(k, i), S(i)) &= 1 - \frac{\hat{\gamma}_i(k)}{A(k, i)} - \\ &\prod_{j \in d_i} \left(1 - \frac{\hat{\gamma}_i(k) \sum_{s \in S(i)} n_j(s, 1)}{A(k, i) \cdot \sum_{s \in S(i)} n_i(s, 1)} \right) = 0 \end{aligned} \quad (6)$$

where

$$\hat{\gamma}_i(k) = \frac{n_i(k, 1)}{n^k}$$

Proof: Assume $k \in S(i)$ sending probes to node i . Based on the first equation of (2), we have

$$A(k, i)\beta_i = \frac{n_i(k, 1)}{n^k} = \hat{\gamma}_i(k), \quad k \in S(i)$$

If $|S(i)| > 1$, there are more than one sources sending probes to node i , the pass rates of two sources, s and $k, s, k \in S(i)$, to node i are correlated that can be expressed as

$$A(s, i) = A(k, i) \frac{\hat{\gamma}_i(s)}{\hat{\gamma}_i(k)}, \quad s, k \in S(i) \quad (7)$$

If there is only one source for node i , (7) can be ignored. Let $n_i^*(1)$ be the total number of probes reaching node i , we have

$$n_i^*(1) = \frac{A(k, i)}{\hat{\gamma}_i(k)} \sum_{s \in S(i)} n^s \hat{\gamma}_i(s)$$

Then, we have two different ways to express $1 - \beta_i$. Using the loss rate of subtree j rooted at node i , we have

$$1 - \beta_i = \prod_{j \in d_i} \left[1 - \frac{\sum_{s \in S(i)} n_j(s, 1)}{n_i^*(1)} \right] \quad (8)$$

and using (5), we have

$$1 - \beta_i = 1 - \frac{\hat{\gamma}_i(k)}{A(k, i)}$$

Connecting the two, we have

$$1 - \frac{\hat{\gamma}_i(k)}{A(k, i)} = \prod_{j \in d_i} \left(1 - \frac{\hat{\gamma}_i(k) \cdot \sum_{s \in S(i)} n_j(s, 1)}{A(k, i) \cdot \sum_{s \in S(i)} n^s \hat{\gamma}_i(s)} \right)$$

Except for $A(k, i)$, all others, e.g. $\hat{\gamma}_i(k), n_j(s, 1)$, are either known or estimable from observations. Thus, the above equation is a polynomial of $A(k, i)$. Alternatively, using (8) to replace β_i from (2), we have the same result. ■

(6) generalizes (3) that considers various paths ending at different nodes, including those having $|S(i)| = 1$ and those having $|S(i)| > 1$. For node i having $|S(i)| = 1$, (6) degrades to (3). For $|S(i)| > 1$, if $j \in d_i$,

$$\frac{\sum_{s \in S(i)} n_j(s, 1)}{\sum_{s \in S(i)} n_i(s, 1)} \quad (9)$$

is the estimate of the pass rate of the link connecting node i to node j , where the numerator is the sum of the probes sent by $S(i)$ that reach $Rs(i)$ while the denominator is the sum of the probes sent by $S(i)$ that reach $Rs(j), j \in d(i)$. Both are obtainable from Ω . The estimate is built on the arithmetic mean that considers the contribution of all sources sending probes to the link.

Solving (6), we have $\hat{A}(k, i)$. Then, we can have $\hat{A}(s, i), s \in S(i) \setminus k$ and

$$\hat{\beta}_i = \frac{n^s}{\hat{A}(s, i)}, s \in S.$$

To prove the uniqueness of (5), we have the following Lemma

Lemma 1. Assume $c_i \in (0, 1)$, if $\sum_i c_i > 1$, $x = \prod_i [(1 - c_i) + c_i x]$ has a unique solution in $(0, 1)$. Otherwise, if $\sum_i c_i < 1$, there is either no solution or have multiple solutions in $(0, 1)$ for the equation.

Proof: See appendix ■

Based on Lemma 1, we have

Theorem 3. The set of likelihood equations formed by (5) has the unique solution.

Proof: Using $\frac{\sum_{s \in S(i)} n_j(s, 1)}{n_i^*(1)}$ to replace c_i in Lemma 1, the theorem follows. ■

To prove the estimate obtained by (6) is the MLE, we resort on a well known theorem for the MLE of a likelihood function yielding the exponential family.

Theorem 4. If a likelihood function belongs to a standard exponential family with $A(s, i)$ as the natural parameters, we have the following results:

- 1) the likelihood equation $\frac{\partial L(\theta)}{\partial \theta_i} = 0$ has at most one solution $\theta_i^* \in \Theta$;
- 2) if θ_i^* exists, θ_i^* is the MLE.

Proof: This theorem can be found from a classic book focusing on exponential families, such as [25]. The likelihood function presented in (6) belongs to the exponential family, where $A(s, i)$ is the natural parameters. Then, the estimate obtained by (6) is unique in its support space and the MLE of $A(s, i)$. ■

IV. SOLUTIONS

Given Theorem 2, there are a number of ways to complete the estimation. One of them is to repeatedly use Theorem 2 to obtain the pass rates of all paths, and then use Γ^{-1} , from path rate to link rate, to have the loss rate of each link, i.e.

$$1 - \theta_i = \frac{\sum_{s \in S(i)} n^s A(s, i)}{\sum_{s \in S(i)} n^s A(s, f(i))} \quad (10)$$

The advantages of this is its uniformity, while the disadvantage could be the requirement of solving a large number of polynomials, some of them may be high degree polynomials.

A. Closed Form Solution

It is easy to see the degree of (6) is one less than the number of descendants connected to node i and we also know that there is no closed form solution for a polynomial that has a degree of 5 or greater. Then, the key to have a closed form solution to (6) is to replace a multi-descendant node, i.e. a node with more than 5 descendants, with a tree. In [23], such a replacement is proposed for the tree topology and called a statistical equivalent replacement since

- it is equal to use a number of serially connected links to replace an egress link of a multi-descendant node, and
- the statistics of the links connecting or connected to the node remain the same as that before the replacement.

Comparing (3) with (6), one can notice that (3) is a special case of (6). Then, the replacement proposed for the tree topology must be a special case of the general topology, where the key rests on how to determine the statistics of the nodes located in the tree used to replace a multi-descendant node. The following theorem extends the result presented in [23] and provides the answer to the above question.

Theorem 5. A multi-descendant node, say i , can be replaced by a tree, where the statistics of the nodes in the tree are determined according to the observation of $R(i)$. The following equation is used for an introduced link z :

$$n_z(1) = \sum_{s \in S(i)} \sum_{j \in d_z} n_{ij}(s, 1) - \sum_{s \in S(i)} \sum_{\substack{i < j \\ i, j \in d_z}} n_{ij}(s, 1) + \sum_{s \in S(i)} \sum_{\substack{i < j < k \\ i, j, k \in d_z}} n_{ijk}(s, 1) - \dots + (-1)^{|d_z|-1} \sum_{s \in S(i)} n_{d_z}(s, 1) \quad (11)$$

where d_z denotes the descendants connected to link z , $n_{ij}(s, 1) = \sum_{u=1}^n (Y_i^u(s) \wedge Y_j^u(s))$ is the number of probes that have been observed by at least one of $R(i)$ and one of $R(j)$ simultaneously, similarly $n_{ijk}(s, 1) = \sum_{u=1}^n (Y_i^u(s) \wedge Y_j^u(s) \wedge Y_k^u(s))$, \dots , and $n_{d_z}(s, 1) = \sum_{u=1}^n (\bigwedge_{j \in d_z} Y_j^u(s))$.

Proof: see appendix. ■

Given Theorem 5, node i can be replaced by a binary tree when we estimate the pass rate of the paths connecting $S(i)$ to node i . Then, equation (6) becomes a linear equation of $A(k, i)$, $k \in S(i)$ and a closed form solution follows.

B. Decomposition

Apart from the closed form solution as above, another two closed form solutions based on divide-and-conquer strategies are proposed as alternatives. This is because equation (6) shows such a fact that the parameter estimation in a hierarchical structure with the probes flowing in one direction can be carried out by dividing the structure into a few segments as the d-separation presented in [26]. For the general topology, given the number of probes reaching a node, the descendants of the node become independent from each other and can be estimated independently. Then, the immediate issue is which nodes should be selected as the decomposing points, where the criterion used to evaluate the strategies is based on the number of independent trees created after a decomposition, the smaller the better since the computation cost of estimation is proportional to the number of independent trees. The following theorem is presented for the optimal strategy to select decomposing points.

Theorem 6. *Given a general network covered by multiple trees, the optimal strategy to decompose the overlapped multiple trees into a number of independent trees is to use the nodes of J as decomposing points.*

Proof: We prove this by contradiction. Firstly, we assume there were two strategies, say A and B, where A decomposes a general network into a number of independent subtrees and there is at least one of the decomposing points that is not a joint point and the total number of independent subtrees is m' , while B decomposes the network at the joint points only and the total number of independent subtrees is m . If strategy A were better than strategy B, we should have $m' < m$. However, this is impossible: given the fact that all of the subtrees rooted at a joint point can only become independent from each other if we know the state of the joint point according to d-separation. Then, we have $m' \geq m + 1$, which contradicts to the assumption. ■

Given $i \in J$ and $n_i^*(1)$ obtained by (6), a network of the general topology can be divided into a number of independent

trees. The independent trees can be divided into two groups on the basis of the position of a tree relative to the decomposing point that separates it from others. We call the two groups: the ancestor group and the descendant one. For the descendant group, given $n_i^*(1)$, the subtrees rooted at node i become independent from their ancestors and from each other. Then, the loss rates of the links in the subtrees can be estimated by assuming a source attached to the root that sends $n_i^*(1)$ probes to the receivers. Further, the maximum likelihood estimators developed for the tree topology, such as the top down algorithm [24], can be applied to each of the subtrees to estimate the loss rate of each link of the subtree.

For the ancestor group, the subtrees are also independent from each other given $n_i^*(1)$. However, given $n_i^*(1)$ but the observations at node i , we need to either use the first equation of (2) or the property of $imp(j, i)$ to estimate the loss rates. The former leads to a modified estimator of (3). For $s, s \in S(i)$, we use $\hat{A}(s, i)$ to replace $\gamma_i(s)$ in the modified path-based estimator that has the following form:

$$1 - \frac{\gamma_{f^s(i)}(s)}{A(s, f^s(i))} = \left[1 - \frac{\hat{A}(s, i)}{A(s, f^s(i))} \right] \prod_{j \in d_{f^s(i)} \setminus i} \left(1 - \frac{\gamma_j(s)}{A(s, f^s(i))} \right). \quad (12)$$

Here, we also need to solve a polynomial to have $A(s, f^s(i))$. Once having $A(s, f^s(i))$, we are able to estimate $n_{f^s(i)}^*(1)$, then we move a level up and use (12) to estimate $A(s, f_2^s(i))$. This process continues from bottom up until reaching s . If the tree being estimated has more than one intersections, at the common ancestors of the intersections, the RHS of (12) will have a number of the left-most terms, one for an intersection plus the product term for those subtrees that do not link to any intersection. If the total number of overlapped trees plus the number of independent subtrees is over 5, there is no direct solution to (12). Compared to the approach of using (6), this approach does not have significant advantage since it also needs to solve a polynomial for a link. This shows the path-based estimators are similar to each other in terms of performance regardless of the topologies.

The question that we are facing is whether there is a more effective approach than the path-based estimators and whether the link-based estimators are better than the path-based one in this circumstance. The answer is affirmative and the solution is reported in [?].

C. Data Consistency

Data consistency was raised in [11] for three extreme circumstances in observation Ω for the tree topology, which are $\hat{\gamma}_i = 0$, $1 - \hat{\theta}_i > 1$, and $\hat{\gamma}_k = \sum_{j \in d_k} \hat{\gamma}_j$. The three extreme circumstances make the estimation of θ_i an impossible task. To continue the estimation without θ_i , three corresponding procedures were proposed to skip the impossible task. For the general topology, the three circumstances still affect the estimation of the links, the paths, and the subtrees that only observe the probes sent by a single source, and the procedures proposed previously are still applicable to the occurrences

of the problems. However, due the the use of multi-sources sending probes in a general topology, the situation is different from the tree topology. We here specify the data consistent issues for the paths and links that can observe probes sent by more than one sources:

- 1) if $\gamma_i(s) = 0, s \in S$, based on the location of link i , we need to consider:
 - if $i \in SBRL$, we remove $T(i)$ except for those that intersects with other trees. The loss rates of those links located in an intersection can still be estimated as long as the probes sent by other sources can reach the receivers attached to the intersection.
 - if $i \in SSNL$, no link will be deleted from the subtree rooted at node i as long as $Rs(i), s \in S(i)$ observes some probes sent by the sources in $S(i)$. The loss rates of the links in $T(i)$ cannot be estimated if no receiver in $Rs(i)$ observes a probe from the sources.
- 2) as [11] stated, $1 - \hat{\theta}_i > 1$ is not a feasible value. An estimator must avoid this circumstance in estimation. The top down algorithm proposed in [24] and the algorithm based on (??) ensure this would not occur by having $\hat{n}_{f(i)}(1) \geq \hat{n}_i(1), \forall i, i \in V$.
- 3) $\hat{\gamma}_i = \sum_{j \in d_i} \hat{\gamma}_j$ was raised in [11]. We call it partition circumstance here because the equation shows that the observations of $R(j), j \in d_i$ are the partitions of the observations of $R(i)$, i.e. the receivers attached to each subtree of $T(i)$ has the exclusive observations for a portion of the probes sent by the source. Given the partition circumstance at node i of a tree topology, equation (3) cannot be used to estimate the loss rate of the links connecting k to its children. For the general topology, the situation is improved if k is a parent of node $j, j \in SSNL$. Even $\hat{\gamma}_k(s) = \sum_{j \in d_k} \hat{\gamma}_j(s)$, we are still able to estimate the loss rate of the link connecting k to j by (6) as long as the partition circumstance does not occur simultaneously for all j 's parents.
- 4) It is clear that (3) is a special case of (6). Each of the three circumstances raised previously for data consistency has its counterpart in the general topology that can be expressed as:
 - a) if $\sum_{s \in S(i)} \gamma_i(s) = 0$,
 - b) if $1 - \theta_i > 1$, and
 - c) if $\sum_{s \in S(i)} n_i(s, 1) = \sum_{j \in d_i} \sum_{s \in S(j)} n_j(s, 1)$.

For the last one, $A(s, i), s \in S(i)$ cannot be determined by (6). If one of the three scenarios occurs, the corresponding procedure proposed in [11] can be used.

As discussed, data consistency has been improved in some degree for the general topology. The improvement is due to the use of multiple sources to send probes that reduces the probability of those extreme circumstances in comparison with the tree topology.

V. STATISTICAL PROPERTY OF THE ESTIMATORS

Apart from proving the three estimators are MLEs, we also study the statistical properties of the estimators, such as whether the estimators are minimum-variance unbiased

estimators (MVUE); and/or the estimate obtained is the best asymptotically normal estimates (BANE), etc. This section provides the results obtained from the study.

A. Minimum-Variance Unbiased Estimator

The estimators proposed in this paper are MVUE and the following theorem proves this.

Theorem 7. *The estimators proposed in this paper are MVUE and the variances of the estimates reach the Carmér-Rao bound.*

Proof: The proof is based on Rao-Blackwell Theorem that states that if $g(X)$ is any kind of estimator of a parameter θ , then the conditional expectation of $g(X)$ given $T(X)$, where T is a sufficient statistics, is typically a better estimator of θ , and is never worse. Further, if the estimator is the only unbiased estimator, then, the estimator is the MVUE.

To prove the estimator is an unbiased estimator, we only consider the estimate of the binary tree since a multi-descendant tree can be transformed into a binary one and equation (6) is an extension of the tree one. For the binary tree, the pass rate of link i , A_i , is estimated by

$$\frac{\hat{\gamma}_1 \hat{\gamma}_2}{\hat{\gamma}_1 + \hat{\gamma}_2 - \hat{\gamma}_i} = \frac{\hat{\gamma}_1 \hat{\gamma}_2}{\hat{\gamma}_{12}} = \frac{n_1(1)n_2(1)}{n * n_{12}(1)}$$

where $\hat{\gamma}_{12} = \frac{n_{12}(1)}{n}$ is the empirical pass rate of the probes that reach both node 1 and node 2. Then, we have

$$\begin{aligned} & E\left(\frac{n_1(1)n_2(1)}{n * n_{12}(1)} - A_i\right) \\ &= E\left(\frac{n_1(1)n_2(1)}{n * n_{12}(1)}\right) - E(A_i) \\ &= E\left(\frac{n_1(1)n_2(1)}{n_i(1)n_{12}(1)} \cdot \frac{n_i(1)}{n}\right) - \frac{1}{n}E\left(\sum_{k=1}^n X_i^k\right) \\ &= \frac{1}{n}E\left(\frac{\sum_{k=1}^{n_i(1)} X_1^k \cdot \sum_{k=1}^{n_i(1)} X_2^k}{n_i(1) \sum_{k=1}^{n_i(1)} X_{12}^k} \cdot \sum_{k=1}^n X_i^k\right) - \frac{1}{n}E\left(\sum_{k=1}^n X_i^k\right) \\ &= \frac{1}{n}E\left(\frac{\sum_{k=1}^{n_i(1)} X_1^k \cdot \sum_{k=1}^{n_i(1)} X_2^k}{n_i(1) \sum_{k=1}^{n_i(1)} X_{12}^k}\right) \cdot E\left(\sum_{k=1}^n X_i^k\right) - \frac{1}{n}E\left(\sum_{k=1}^n X_i^k\right) \\ &= \frac{1}{n}E\left(\sum_{k=1}^n X_i^k\right) - \frac{1}{n}E\left(\sum_{k=1}^n X_i^k\right) = 0 \end{aligned} \quad (13)$$

The statistics used in this paper has been proved to be the minimal complete sufficient statistics. Based on Theorem 3, we have a unique solution in $(0, 1)$ for the parameter. Then, applying Rao-Blackwell theorem, the theorem follows.

Given theorem 7, it is easy to prove the variance of the estimates, e.g. $\hat{\theta}_i$ and $\hat{\beta}_i$, obtained by (6) are equal to Carmér-Rao low bound since (6), the likelihood function leading to (6), belongs to the standard exponential family. ■

Based on Fisher information we can prove the variance of the estimates obtained by (6) from Ω is also smaller than that of an estimate obtained by (3) from $\Omega_s, s \in S$. Since

the receivers attached to intersections observe the probes sent from different sources, the sum of the observed probes is at least larger than or equal to the maximum number of probes observed from a single source. Therefore, there is more information about the loss rates of the links located in the intersections since information is additive under *i.i.d.* assumption. With more information, the variance of an estimate of a parameter must be smaller than another obtained from the probes sent by a single source according to Fisher information. Given the less varied estimates from the intersections, the variances of the estimates of other links that are not in intersections are also reduced, at least not increased. Therefore, the estimates obtained by (6) is better than those obtained from a single source.

B. General Topology vs. Tree Topology

The results presented in this paper can be viewed as a generalization of the results presented in [11] since the tree topology is only a special case of the general topology. The findings and discovery presented in this paper cover those presented in [11]. The following corollary confirms this:

Corollary 1. *Any discovery, including theorems and algorithms, for loss estimation in the general topology, holds for the tree topology as well.*

For instance, (6) obtained for the general topology holds for the tree one. When $S(i) = \{k\}$, we have

$$\begin{aligned} H(A(s, k), S(k)) &= 1 - \frac{\gamma_i(k)}{A(k, i)} - \\ &\prod_{j \in d_i} \left(1 - \frac{\gamma_i(k) \sum_{s \in S(i)} n_j(s, 1)}{A(k, i) \cdot \sum_{s \in S(i)} n_i(s, 1)} \right) \\ &= 1 - \frac{\gamma_i(k)}{A(k, i)} - \prod_{j \in d_k} \left(1 - \frac{\gamma_j(k) n_j(1)}{A(k, i) \cdot n_i(1)} \right) \\ &= 1 - \frac{\gamma_i(k)}{A(k, i)} - \prod_{j \in d_k} \left(1 - \frac{\gamma_j(k)}{A(k, i)} \right) = 0 \end{aligned} \quad (14)$$

the last equation is $H(A_i, i)$ presented in [11]. Another example has been presented in Section IV-C 4 for data consistency.

On the other hand, we are also interested in whether those properties discovered from the tree topology can be extended into the general one and the difference between the original properties and the extended ones, in particular for the rates of convergence.

C. Large Sample Behavior of the Estimator

Since the estimate obtained by the proposed estimator is the MLE $\hat{\theta}_i$, we are able to apply some general results on the asymptotic properties of MLEs in order to show that $\sqrt{n}(\hat{\theta}_i - \theta_i)$ is asymptotically normally distributed as $n \rightarrow \infty$. Using this, we can estimate the number of probes required to have an estimate with a given accuracy for many applications. The fundamental object controlling convergence rates of the MLE is the Fisher Information Matrix at θ_i . Since $L(\theta)$ yields exponential family, it is straightforward to verify that $\hat{\theta}_i$ is

consistent and that $L(\theta)$ satisfies conditions under which I is equal to

$$I_{jk}(\theta) = -E \frac{\partial^2 L}{\partial \theta_j \partial \theta_k}(\theta)$$

Eliminate singular on the boundary of $(0, 1]^{|E|}$, we have

Theorem 8. *When $\theta_i \in (0, 1), i \in V \setminus S, \sqrt{n}(\hat{\theta}_i - \theta_i)$ converges in distribution as $n \rightarrow \infty$ to an $|E|$ dimensional Gaussian random variable with mean 0 and covariance matrix $I^{-1}(\theta)$, i.e.*

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} N(0, I^{-1}(\theta))$$

and $\hat{\theta}_i$ is the best asymptotically normal estimate (BANE).

Proof: It is known that under following regularity conditions:

- the first and second derivatives of the log-likelihood function must be defined.
- the Fisher information matrix must not be zero, and must be continuous as a function of the parameter.
- the maximum likelihood estimator is consistent.

the MLE has the characteristics of asymptotically optimal, i.e., asymptotically unbiased, asymptotically efficient, and asymptotically normal. The characteristics are also called BANE.

It is clear that (6), the likelihood function used in this paper, belongs to the standard exponential family, which ensures the consistency and uniqueness of the MLE. To satisfy the second condition for the exponential family, $n_i(s, 1), i \in E, s \in S$ should not be linearly related, this is true as $n_i(s, 1), i \in E, s \in S$ have been proved to be the minimal sufficient statistics, see Theorem 2 of [23]. Then, we only need to deal with the first condition. Obviously, (2) has both first and second derivatives in $(0, 1)^{|E|}$, and $L(\theta)$ is strictly concave, which ensures the Fisher information matrix $I(\theta)$ positive definiteness. ■

Theorem 8 states such a fact that with the increase of the number of probes sent from sources, there are more probes reaching the links of interest. Then, there is more information for the paths to be estimated. Let $I_0(\theta)$ is the Fisher Information for a single observation, we have $I(\theta) = nI_0(\theta)$, where n is the number of observations related to the link/path being estimated.

As $n \rightarrow \infty$, the difference in terms of Fisher information between the two estimators approaches to zero. Therefore, as $n \rightarrow \infty$, estimation can be carried out on the basis of individual tree and the asymptotical properties obtained previously for the tree topology [11] hold for the general topology as well. On the other hand, if $n < \infty$, the estimate obtained by (6) is more accurate than those obtained from an individual tree. The simulation results presented in the next section illustrate this that shows the fast convergence of the estimates for the links located in the intersection because there are more information about the links.

Although there are a number of large sample properties that can be related to loss inference in the general topology, including various asymptotic properties, we would not discuss them further because $n \rightarrow \infty$ means infinite number of probes

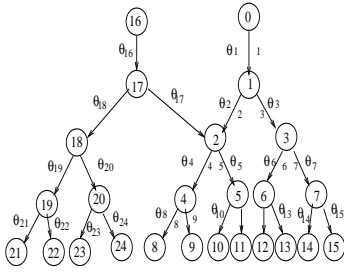


Fig. 1. A Network with Multiple Sources

be sent from sources to receivers that requires a long period of stationarity of the network, including traffic and connectivity, which is impractical based on the measurement [27].

VI. SIMULATION STUDY

For the purpose of proof of concept, a series of simulations were conducted on a simulation environment built on *ns2*, the network simulator 2 [28], the network topology used in the simulation is shown in Figure 1, where two sources located at node 0 and node 16 multicast probes to the attached receivers in an interval of 0.01 second. The binary network used here is for the simplicity reasons since no matter how many branches a node has they are always converted to a two-layer binary tree in estimation. Apart from the traffic created by probing, a number of TCP sources with various window sizes and a number of UDP sources with different burst rates and periods are added at the roots and internal nodes to produce cross-traffic, where TCP traffic takes about 80% of the total traffic. The loss rate of each link is controlled by a random process that has 1% drop rate except link 8 that has 10% drop rate. Each simulation run for 200 simulation seconds, and executed under the same condition except for using different seeds. The samples collected in this period are divided into groups to study the effect of group size on the accuracy of estimation. Five groups for 200, 400, 600, 800, and 1000 samples are presented here to illustrate the impact of the group sizes on the accuracy of estimation. The accuracy is measured by the relative error that is defined as:

$$\frac{\text{abs}(\text{actual loss rate} - \text{estimated loss rate})}{\text{actual loss rate}}$$

where we use *ns2* to report the losses and passes incurred on individual links. The ratio between the losses and the sum of the losses and passes is used as actual loss rate. The estimated loss rate of a link is obtained by only considering the end-to-end observations. The resultant relative errors of the 24 links are presented in Figure 2 to Figure 4, each for 8 links. The figures show that in general with the increase of the group size, the relative errors are decreased as expected. Although there are a few cases of slight increase in the figures, that is due to the randomness of relative small group sizes used in estimation. The problem can be overcome with the increase of the group sizes. For example, observing Figures 2 and 3, one is able to notice the consistent decrease of the relative errors with increase of the group size for those links falling into the intersection, i.e., link 4, 5, and 8-11 because there are

almost 2 times of probes traversing through this intersections in comparison with other links. The relative errors of the shared links drop below 15% when each source sends 1000 probes; when 2000 probes are sent by each source, the relative errors of those links drop below 10% (not presented here). This phenomenon suggests the use of large group size in estimation for high accuracy. However, whether we are able to use more samples in estimation is a question that is not only related to the complexity of an algorithm but also related to the time scale of the stationarity of the network of interest. Based on the setup of our simulation, a 20-second (0.01×2000) stability is needed to have the relative error drops to 10%. If the time scale of the stability of a network is much larger than 20 seconds, the estimates obtained by the proposed estimators have potential to be used in traffic engineering to control traffic flows.

From Figures 2 and 3, there is a noticeable difference between link 8 and link 9 in their converging rate in terms of the relative error. The two links are siblings, that means the same number of probes observed by their common parent first before sending to receiver 8 and 9. However, the relative error of link 8 converges much quick than that of link 9, while link 8 has a much higher loss rate than link 9 does. This indicates that the accuracy of estimating the loss rate of a link is proportional to the number of probes observed by its siblings. This can be explained analytically. For instance, given a 2 layer binary tree with 4 nodes, node 0 is the root that connects to node 1, node 1 has two children 2 and 3 and uses link 2 and 3 to connect them; the MLE of the loss rate of link 3 is equal to

$$\hat{\theta}_3 = \frac{n_3(0)}{n_2(1)}$$

[29]. In this simulation, the loss rate of link 8 is set to 10% that is 10 times of that of link 9, which makes $n_8(1)$ much smaller than $n_9(1)$ and vice versa for $n_8(0)$ and $n_9(0)$. This implies that we get more information for link 8 than that for link 9 from observations. That is why the relative error of link 9 is higher than that of link 8. This leads to a finding that extends the finding of [11] on branching ratio where the authors stated that estimate variance reduces with the increase of branching ratio. The fact of increasing branching ratio is equal to the decrease of the loss rate of the siblings of a link being estimated, that leads to more information about the link from the observations of the siblings. Therefore, to have an accurate estimate of a link, apart from increasing the number of samples in a group we can increase the number of siblings of a link instead. In contrast to previous algorithms, the increase of siblings would not significantly affect the complexity of the algorithms proposed in this paper since they all have $O(|E|)$ complexity.

VII. CONCLUSION

In this paper, we presents our recent findings on loss tomography of the general topology, including both theoretical results and practical algorithms. In theory, we extend the set of complete minimal sufficient statistics proposed previously for the link-based estimator into the path-based estimator. Based on the statistics, we provide the path-based likelihood

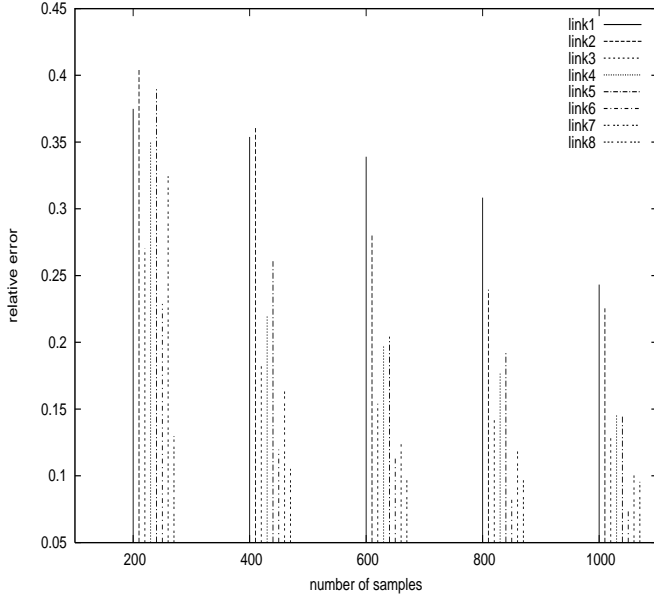


Fig. 2. Relative Error for Link 1-8

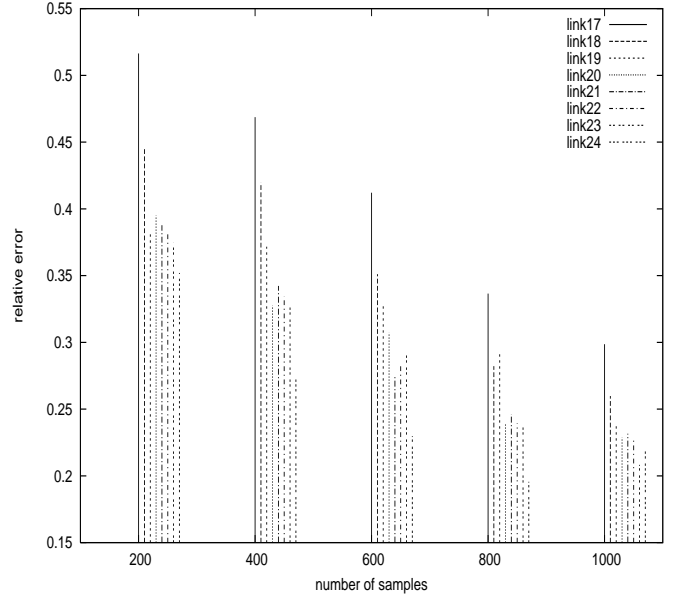


Fig. 4. Relative Error for Link 17-24

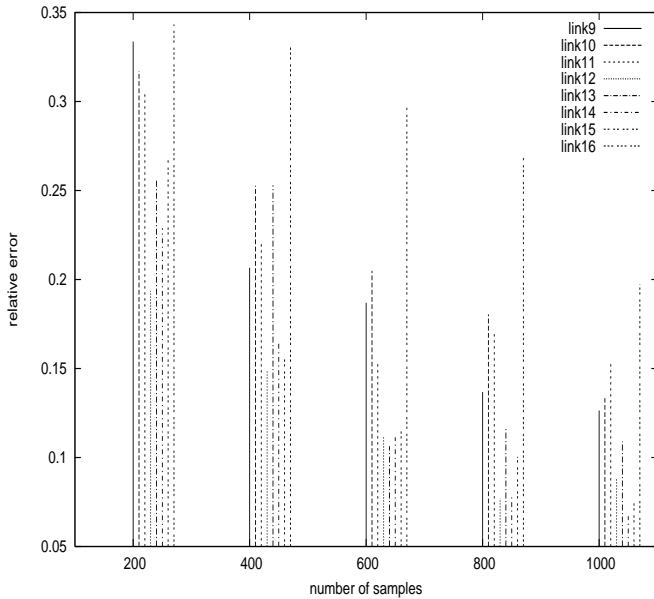


Fig. 3. Relative Error for Link 9-16

function and have a set of likelihood equations. Solving the likelihood equations, we derive a direct expression of the MLE for the pass rate of the paths in a general topology. The direct expression has a similar structure as its counterpart in the tree topology, a polynomial is for a path. The expression in the general topology differs from the one in the tree topology in the collective constraint, where a set of polynomials is bounded together if the paths end at the same node that has a descendant tree. In this aspect, the polynomial derived for the tree topology [11] can be regarded as a special case of the one derived in this paper.

With the direct expression, we are able to estimate the

number of probes reaching the root of an intersection. Given the estimated number of probes reaching the roots of all intersections, the network can be decomposed into a number of independent trees. For the descendant trees, the algorithms proposed for the tree topology can be applied to them to estimate the loss rates of the links. For the ancestor trees that are not from the intersections, new algorithms are needed since there is no exact observations for the leaf nodes that are the joint nodes before decomposition. Two types of algorithms, path-based and link-based, are presented and compared in this paper. The result shows that the link-based one is significantly better than the path-based one here in terms of performance.

The method proposed in [23] to merge subtrees rooted at the same link has been extended into the general topology to merge the statistics obtained from multiple sources. Using this method, we are able to reduce the degree of the polynomials used to express the pass rate of a path, and subsequently avoid the use of a numeric method to solve a high degree of polynomial. The estimator proposed in this paper consists of 3 steps

- 1) using (6) to estimate the pass rates of the paths connecting sources to joint nodes,
- 2) using the top down algorithm to estimate the loss rates of the links in descendant trees, and
- 3) using (??) to estimate the loss rate of the links in ancestor trees.

If no complicated overlapping occurs between the trees used to cover a network, there is no need to have any iterative procedures involved in estimation. The complexity of the estimator is $O(|E|)$. This performance is significantly better than the iterative algorithms, including the EM.

Data consistency raised previously for the tree topology [11] has been reconsidered for the general topology. Our study shows this problem remains but has been eased when multiple

sources are used to send probes to receivers. In addition, our study shows the convergence rate of the links that receive probes from more sources is faster than those that only receive from one.

APPENDIX

Lemma 1

Proof: Let $h_1(x) = x$ and $h_2(x) = \prod_i [(1 - c_i) + c_i x]$, we have $h'_1(x) = 1$ and $h'_2(x) = h_2(x) \sum_i \frac{c_i}{(1 - c_i) + c_i x}$. Let $q_i = \sum_i \frac{c_i}{(1 - c_i) + c_i x}$, we have $h''_1(x) = 0$ and $h''_2(x) = h_2(x) [(\sum_i q_i)^2 - \sum_i q_i^2] > 0$, if $x \in [0, 1]$. Let $h(x) = h'_1(x) - h'_2(x)$, that is strictly concave on $[0, 1]$. In addition, $h'(0) = 1 - \prod_i (1 - c_i) \sum_i \frac{c_i}{1 - c_i} > 0$ and $h'(1) = 1 - \sum_i c_i$. If $\sum_i c_i > 1$, $h'(1) < 0$, there is a unique solution to $h(x) = 0$ for $x \in [0, 1]$ since $h(x)$ is continuously differentiable on $[0, 1]$, and $h(0) = -\prod_i (1 - c_i) < 0$ and $h(1) = 0$. Otherwise, if $\sum_i c_i < 1$, $h'(1) > 0$, there is no solution to $h(x) = 0$ for $(0, 1)$. ■

Theorem 5

Proof: It has been proved that each multicast tree or subtree can be treated as a virtual link in estimation [11]. Here, we want to prove that a number of virtual links rooted at the same node can be considered a virtual link in estimation if the statistics of the subtrees are merged according to the theorem. To prove such a transformation is an equivalent one, we need to prove the following

- 1) the statistics of node i remains the same after the transformation.
- 2) the estimate of θ_i obtained from the internal view of the transformation is equal to that from the internal views of the origin.

It is known that $n_i(s, 1) = \sum_{l=1}^{n(s)} (\bigvee_{p \in d_i} Y_p^l(s))$. If we divide d_i into d_{i1} and d_{i2} and let $Y_{d_{ik}}^l(s) = \bigvee_{p \in d_{ik}} Y_p^l(s)$, $k \in \{1, 2\}$ be the observation of d_{ik} for probe l sent by source s , $\sum_{s \in S(i)} n_i(s, 1) = \sum_{s \in S(i)} \sum_{l=1}^{n(s)} (Y_{d_{i1}}^l(s) \bigvee Y_{d_{i2}}^l(s))$ is the statistics of d_{ik} . If d_{ik} , $k \in \{1, 2\}$ is merged into a virtual link and node ik is assumed to be the child node of the virtual link, then

$$\begin{aligned} n_{ik}(1) &= \sum_{s \in S(i)} \sum_{l=1}^{n(s)} \bigvee_{j \in d_{ik}} Y_j^l(s) \\ &= \sum_{s \in S(i)} \left[\sum_{j \in d_{ik}} n_j(s, 1) - \sum_{\substack{i < j \\ i, j \in d_{ik}}} n_{ij}(s, 1) + \right. \\ &\quad \left. \sum_{\substack{i < j < k \\ i, j, k \in d_{ik}}} n_{ijk}(s, 1) - \dots + (-1)^{|d_{ik}|-1} n_{d_{ik}}(s, 1) \right] \end{aligned}$$

Then, 1) holds.

Given $n_{ik}(1)$, $k \in \{1, 2\}$, the RHS of (9) can be written as

$$\begin{aligned} RHS &= \left(1 - \frac{n_{i1}(1)}{A(k, i) \cdot \sum_{s \in S(i)} n_s(1) \hat{\gamma}_i(s)} \right) \cdot \\ &\quad \left(1 - \frac{n_{i2}(1)}{A(k, i) \cdot \sum_{s \in S(i)} n_s(1) \hat{\gamma}_i(s)} \right) \quad (15) \end{aligned}$$

To avoid the partition problem, when we select subtrees to merge, we need to ensure the observations of the two group receivers are intersected. Then, the theorem follows. ■

REFERENCES

- [1] Y. Vardi. Network Tomography: Estimating Source-Destination Traffic Intensities from Link Data. *Journal of Amer. Stat. Assoc.*
- [2] F. LoPresti & D. Towsley N.G. Duffield, J. Horowitz. Multicast topology inference from measured end-to-end loss. *IEEE Trans. Inform. Theory*, 48, Jan. 2002.
- [3] G. Liang and B. Yu. Maximum pseudo likelihood estimation in network tomography. *IEEE trans. on Signal Processing*, 51(8), 2003.
- [4] Y. Tsang, M. Coates, and R. Nowak. Network delay tomography. *IEEE Trans on Signal Processing*, (8), 2003.
- [5] F.L. Presti, N.G. Duffield, J. Horowitz, and D. Towsley. Multicast-based inference of network-internal delay distribution. *IEEE/ACM Trans. on Networking*, (6), 2002.
- [6] Meng-Fu Shih and Alfred O. Hero III. Unicast-based inference of network link delay distributions with finite mixture models. *IEEE Trans on Signal Processing*, (8), 2003.
- [7] E. Lawrence, G. Michailidis, and V. Nair. Network delay tomography using flexicast experiments. *Journal of Royal Statist. Soc.*, (Part5), 2006.
- [8] V. Arya, N.G. Duffield, and D. Veitch. Multicast inference of temporal loss characteristics. *Performance Evaluation*, (9-12), 2007.
- [9] G. Liang, N. Taft, and B. Yu. A fast lightweight approach to origin-destination ip traffic estimation using partial measurements. *IEEE/ACM trans. on Networking*, 14(6), 2006.
- [10] D. Rubenstein, J. Kurose, and D. Towsley. Detecting Shared Congestion of Flows Via End-to-End Measurement.
- [11] R. Cáceres, N.G. Duffield, J. Horowitz, and D. Towsley. Multicast-based inference of network-internal loss characteristics. *IEEE Trans. on Information Theory*, 45, 1999.
- [12] N. Duffield, J. Horowitz, F. Presti, and D. Towsley. Explicit loss inference in multicast tomography. *IEEE Trans. on Information Theory*, 52(8), Aug., 2006.
- [13] W. Zhu and K. Deng. A top down approach to estimate loss rate. In *Proc of IEEE ChinaCom 2006, October 25-27, Beijing, China*, 2006.
- [14] A. Dempster, N. Laird, and D. Rubin. Maximum likelihood from incomplete data via the EM algorithm. *Journal of the Royal Statistical Society*, B, 1977.
- [15] T. BU, N. Duffield, F.L. Presti, and D. Towsley. Network Tomography on General Topologies. In *Proc. of ACM SIGMETRICS 02*, 2002.
- [16] R. Cáceres, N.G. Duffield, S.B. Moon, and D. Towsley. Inference of Internal Loss Rates in the MBone. In *IEEE/ISOC Global Internet'99*, 1999.
- [17] R. Cáceres, N.G. Duffield, S.B. Moon, and D. Towsley. Inferring link-level performance from end-to-end multicast measurements. Technical report, University of Massachusetts, 1999.
- [18] C.R. Rao. Discussion – minimum chi-square, not maximum likelihood. *The Annals of Statistics*, (3), 1980.
- [19] K. Harfoush, A. Bestavros, and J. Byers. Robust identification of shared losses using end-to-end unicast probes. In *Technical Report BUCS-2000-013*, Boston University, 2000.
- [20] M. Coates and R. Nowak. Unicast network tomography using EM algorithms. Technical Report TR-0004, Rice University, September 2000.
- [21] W. Zhu and Z. Geng. A fast method to estimate loss rates. *Lecture Note in Computer Science*, Vol. 3090/2004, 2004.
- [22] W. Zhu and Z. Geng. A bottom up inference of loss rate. *Computer Communications*, 28, 2005.
- [23] W. Zhu and K. Deng. Loss tomography from tree topology to general topology. *Submitted for Publication*, 2009.
- [24] W. Zhu. Loss rate estimation in general topologies. In *Proc. of IEEE Communications Society/CreateNet BROADNETS, San Jose, CA, USA*, 2006.
- [25] L. Brown. *Fundamentals of Statistical Exponential Families*. Institute of Mathematical Statistics, Hayward CA, 1986.
- [26] J. Pearl. *Causality*. Cambridge University Press, 2000.
- [27] V. Paxson. End-to-End Internet Packet Dynamics. *IEEE/ACM Trans. on Networking*, 7(3), 1999.
- [28] The network simulator 2. Technical report, www.isi.edu/nsnam/ns2.
- [29] W. Zhu. Pseudo maximum likelihood loss estimates for general topologies. In *Proc. of IEEE GLOBECOM 2006, November 27- December 1. San Francisco, USA*, 2006.